PROPAGATION OF DISCONTINUITIES IN SOUND WAVES

(RASPROSTRANENIE RAZRYVOV V ZBUKOVYKH VOLNAKH)

PMM Vol.22, No.4, 1958, pp.561-564

K. E. GUBKIN
(Moscow)

(Received 4 March 1958)

In this paper a solution of the equation of gas dynamics is presented which describes the propagation of waves of small amplitude. The extent of the disturbed region (the length in the direction of motion of the wave front within which the variables in the disturbed motion are of significant magnitude) is assumed to be small in comparison with the characteristic dimension of the problem. The nonlinear properties of the motion, exhibited in effect whenever the wave traverses distances substantially greater than the length of the disturbed region, leads to a change in the profile of the wave, and the development of discontinuities in it. Computation of the nonlinear factors points to a damping of the shock front, a result that is in accord with acoustic theory [1, 2]. In the case of propagation of spherical (or cylindrical) waves in a homogeneous stationary medium, the results agree with those obtained earlier in work by Landau [3], Khristianovich [4], and Whitham [5].

1. Characteristics. The basic system of equations of motion of a compressible gas has the form:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_k}\rho v_k=0,\quad \rho\frac{\partial v_i}{\partial t}+\rho v_k\frac{\partial v_i}{\partial x_k}+\frac{\partial p}{\partial x_i}=\rho g_i,\quad \frac{\partial p}{\partial t}+v_k\frac{\partial p}{\partial x_k}-c^2\frac{\partial\rho}{\partial t}-c^2 v_k\frac{\partial\rho}{\partial x_k}=0 \tag{1.1}$$

where v_i, ρ, p, c denote respectively the velocity of the gas density, pressure, and velocity of sound, at a point with Cartesian coordinates x_i at time t; g_i is the acceleration due to gravity, and the indices i and k take the values 1, 2 and 3. The pressure, density and velocity of sound are related to one another by the equation of state $\gamma p = \rho c^2$ where γ is a constant equal to the ratio of the specific heats of the gas.

We consider the characteristics of the system (1.1). The equation determining the characteristic surfaces $\phi(x_i, t)$ = constant has the form:

$$\left[\left(\frac{\partial\varphi}{\partial t}+v_k\frac{\partial\varphi}{\partial x_k}\right)^2-c^2\left(\frac{\partial\varphi}{\partial x_k}\right)^2\right]\left(\frac{\partial\varphi}{\partial t}+v_k\frac{\partial\varphi}{\partial x_k}\right)^3=0 \tag{1.2}$$

If the expression in square brackets in equation (1.2) is set equal to zero, we obtain

$$\frac{\partial\varphi}{\partial t} + v_k \frac{\partial\varphi}{\partial x_k} \pm c\sqrt{\left(\frac{\partial\varphi}{\partial x_k}\right)^2} = 0 \tag{1.3}$$

which determines two families of characteristics C_+ and C_- (corresponding to the choice of signs of the square root). If in the equation $\phi(x_i, t)$ = constant, t is considered a parameter, in place of characteristic surfaces C_+ and C_- stationary in (x_i, t) space, we would have moving surfaces N_+ and N_- in (x_i) space. The velocity of propagation of these surfaces, according to equation (1.3), is equal to $v_i \pm cn_i$ where n_i, the unit vector normal to the surfaces N_+ or N_-, is defined by

$$n_i = \frac{\partial\varphi / \partial x_i}{\sqrt{(\partial\varphi / \partial x_k)^2}}$$

Setting the second bracket of equation (1.2) equal to zero, we obtain

$$\frac{\partial\varphi}{\partial t} + v_k \frac{\partial\varphi}{\partial x_k} = 0 \tag{1.4}$$

which determines a family of characteristic surfaces C_0, defining the paths of particles. Corresponding to the characteristic C_0, moving in (x_i) space, we now have a surface S moving with the gas particles.

We use the following notation for the derivative along a characteristic surface,

$$\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} - \frac{\partial\varphi / \partial x_i}{\partial\varphi / \partial t}\frac{\partial}{\partial t}$$

As a result of the reduction of the system (1.1) to characteristic form, we obtain equations containing derivatives of the desired functions only along corresponding characteristic surfaces. Along the characteristics C_+ and C_-

$$(v_k \pm cn_k)\left(\frac{\partial p}{\partial x_k}\right) \pm \rho c\,(n_i v_k \pm c\delta_{ik})\left(\frac{\partial v_i}{\partial x_k}\right) = \pm\,\rho c n_k g_k \tag{1.5}$$

The upper sign in (1.5) corresponds to C_+, and the lower one to the C_- characteristic; $\delta_{ik} = 1$ for $i = k$, and $\delta_{ik} = 0$ for $i \neq k$. Along the C_0 - characteristic

$$\rho s_i v_k\left(\frac{\partial v_i}{\partial x_k}\right) + s_i\left(\frac{\partial p}{\partial x_i}\right) = \rho g_i s_i, \qquad v_k\left(\frac{\partial p}{\partial x_k}\right) - c^2 v_k\left(\frac{\partial \rho}{\partial x_k}\right) = 0 \tag{1.6}$$

The vector s_i in (1.6) satisfies the condition $s_i\partial\phi / \partial x_i = 0$ (that is, the direction of s_i is tangential to the stream surface S). Since on the surface S there are two independent vectors satisfying this condition, the first equation (1.6) resolves into two independent equations.

Thus, along the C_0-characteristic, we have three relations, corresponding to the fact that for a hyperbolic system of equations (1.1), surfaces determined by the trajectories of particles are three coincident families of characteristics.

2. Geometrical acoustics. Propagation of discontinuities of small amplitude. We determine the solution of characteristic systems of equations which correspond to the propagation of waves of small amplitude with a disturbed region of small extent.

Suppose that in the undisturbed region, the pressure p_0, density ρ_0 and the fluid velocity U_i, are independent of time, and given as functions of the coordinates. We assume that the disturbance pressure $\Delta = p - p_0$, density $\delta = \rho - \rho_0$ and velocity $u_i = v_i - U_i$ are small quantities of the first order in comparison with the undisturbed pressure p_0, density ρ_0 and velocity of sound c_0. The length of the disturbed region λ is assumed to be small in comparison with the radius of curvature R of the wave front, and in comparison with the characteristic length H in which the medium changes perceptibly. In this case it is necessary to neglect terms of the order Δ^2, $\lambda\Delta/R$, $\lambda\Delta/H$ in comparison with Δ. Furthermore we shall assume that in directions tangential to the wave front, the changes in magnitude of the functions Δ, δ, u_i are of the same order as the quantities themselves over distances much larger than the length of the disturbed region (that is, over distances of the order of R and H).

We consider a diverging wave bounded externally by the surface N. We determine the C_+ characteristics so that at the initial instant $t = 0$ they pass through a family of surfaces equidistant from the wave front (N_+ for $t = 0$). The characteristics C_- and C_0 are determined so that they pass through the wave front at the same time. With this choice of characteristics, the angles between the normals to the moving surfaces N_+, N_-, S, N, within the disturbed region of length of order λ, will not exceed magnitudes of the order λ/R, λ/H.

We write equation (1.6) for the C_0-characteristics in the form

$$\frac{d}{dt}(u_k s_k) = u_k \frac{ds_k}{dt} - \frac{s_k}{\rho}\left(\frac{\partial \Delta}{\partial x_k}\right) - s_i u_k \frac{\partial U_i}{\partial x_k} + \frac{\delta}{\rho\rho_0} s_k \frac{\partial p_0}{\partial x_k}$$

$$\frac{d}{dt}(\Delta - c^2\delta) = (c^2 - c_0^2)\, U_k \frac{\partial \rho_0}{\partial x_k} - u_k\left(\frac{\partial p_0}{\partial x_k} - c^2 \frac{\partial \rho_0}{\partial x_k}\right) - \delta \frac{dc^2}{dt}$$

Here $d/dt = v_k(\partial/\partial x_k)$ indicates differentiation along a particle path.

Integrating these equations with respect to time along the trajectory of the particle, and noting that the time a particle remains in the disturbed region is λ/c_0, we obtain

$$s_k u_k = O(\lambda\Delta / R) + O(\lambda\Delta / H) \tag{2.1}$$
$$\Delta - c_0{}^2\delta = O(\lambda\Delta / R) + O(\lambda\Delta / H) + O(\Delta^2) \tag{2.2}$$

The first of these equations is equivalent to:

$$u_i - un_i = O_i(\lambda\Delta / R) + O_i(\lambda\Delta / H) \qquad (u = \sqrt{u^2{}_k})$$

The equation in the C_--characteristics may be written in the form

$$\frac{d}{dt}(\Delta - \rho cu) = \rho cn_i u_k \frac{\partial U_i}{\partial x_k} - \gamma\Delta\frac{\partial U_k}{\partial x_k} - \rho c^2 u \frac{\partial n_k}{\partial x_k} -$$
$$-\left(u_k + \frac{c\delta}{\rho_0} n_k\right)\frac{\partial p_0}{\partial x_k} + \rho c(n_i v_k - c\delta_{ik})\left(\frac{\partial}{\partial x_k}\right)(u_i - un_i) - u\frac{d\rho c}{dt}$$

Here $d/dt = (v_k - cn_k)(\partial/\partial x_k)$ signifies the derivative along the trajectory of the element of a surface N_-.

Since the surface N_- is found in the disturbed region for a time λ/c_0, integration of the equation with respect to t gives

$$\Delta - \rho_0 c_0 u = O(\lambda\Delta / R) + O(\lambda\Delta / H) + O(\Delta^2) \tag{2.3}$$

We notice that equations (2.1), (2.2) and (2.3) agree with relations at the shock front, if small terms of order $\lambda\Delta/R$, $\lambda\Delta/H$, Δ^2, are neglected; therefore the shock front does not influence the flow behind it to that approximation.

We now turn to the equation for the C_+ characteristics (1.5), which we write in the form

$$2\frac{d\Delta}{dt} - \frac{\Delta}{\rho c}\frac{d\rho c}{dt} + \Delta\left(c\frac{\partial n_k}{\partial x_k} + \gamma\frac{\partial U_k}{\partial x_k} + n_i n_k \frac{\partial U_i}{\partial x_k}\right) =$$
$$= \rho c(n_i v_k + c\delta_{ik})\left(\frac{\partial}{\partial x_k}\right)\left(\frac{\Delta}{\rho c} n_i - u_i\right) + \left(\frac{c\delta}{\rho} n_k - u_k\right)\frac{\partial p_0}{\partial x_k} + n_i(\Delta n_k - \rho c u_k)\frac{\partial U_i}{\partial x_k}.$$

Here $d/dt = (v_k + cn_k)(\partial/\partial x_k)$ denotes the derivative along the ray - the trajectory of an element of the surface N_+. Integrating this equation along the ray, and neglecting small terms of order $\lambda D/R$, $\lambda\Delta/H$, and Δ^2 in comparison with Δ, we obtain the result:

$$\Delta = a\frac{\sqrt{\rho_0 c_0}}{L}, \qquad L = \exp\left[\frac{1}{2}\int_0^t\left(c_0\frac{\partial n_k}{\partial x_k} + \gamma\frac{\partial U_k}{\partial x_k} + n_i n_k\frac{\partial U_i}{\partial x_k}\right)dt\right] \tag{2.4}$$

The magnitude a (on the given ray) depends on the size of the surface N_+. If the length of the ray is designated by l, the velocity of motion of the surface N_+ is dl/dt, and therefore the equation for the C_+-characteristic (1.3) can be written along the ray in the form

$$\frac{dl}{dt}=\sqrt{(v_k+cn_k)^2}=$$
$$=\sqrt{(U_k+c_0n_k)^2}+\frac{n_kU_k+c_0}{\sqrt{(U_k+c_0n_k)^2}}\frac{\gamma+1}{2}\frac{\Delta}{\rho_0c_0}+O\left(\Delta\frac{\lambda}{R}\right)+O\left(\Delta\frac{\lambda}{H}\right)+O(\Delta^2)$$

If, in this equation, we replace Δ by its value from equation (2.4), and neglect terms which are small compared to Δ, we obtain, on integration, the solution

$$l-l_0=\int_0^t\sqrt{(U_k+c_0n_k)^2}\,dt+\alpha\frac{\gamma+1}{2}\int_0^t\frac{(n_kU_k+c_0)\,dt}{\sqrt{(U_k+c_0n_k)^2}\sqrt{\rho_0c_0}\,L}+f(\alpha) \qquad (2.5)$$

where $f(\alpha)$ is an arbitrary function which is determined by the pressure specified in the wave at the initial time, l_0 is the position of the wave front for $t=0$. The first term on the right of the solution (2.5) determines the translational motion of the wave as a whole; the second determined the change of its "profile" with time.

We now obtain the equation for a ray in a similar manner. In equation (1.3), which determines the motion of a N_+ surface, we may neglect terms which are small in comparison with the undisturbed velocity of sound c_0, and so determine the ray from the equation

$$\frac{d\varphi}{\partial t}+U_k\frac{\partial\varphi}{\partial x_k}+c_0\sqrt{\left(\frac{\partial\varphi}{\partial x_k}\right)^2}=0$$

From this, we have the equation of the ray in the form

$$\frac{dx_i}{dt}=U_i+c_0n_i,\qquad \frac{dn_i}{dt}=(n_in_k-\delta_{ik})\left(\frac{\partial c_0}{\partial x_k}+n_j\frac{\partial U_j}{\partial x_k}\right)$$

We can use the solution of (2.5) which we obtained earlier, to determine the law of change of pressure at the front of the shock wave. Along the path of the front $l=l(t)$ we have

$$\frac{dl}{dt}=\sqrt{(U_k+c_0n_k)^2}+\frac{n_kU_k+c_0}{\sqrt{(U_k+c_0n_k)^2}}\frac{\gamma+1}{4}\frac{\Delta}{\rho_0c_0} \qquad (2.6)$$

which follows directly from the equation relating the pressure at the shock front with its velocity, if only terms of the first order are considered. Differentiating equation (2.5) with respect to α along the path of the front, and replacing dl/dt by its value from (2.6), we obtain

$$\alpha\frac{n_kU_k+c_0}{\sqrt{(U_k+c_0n_k)^2}\sqrt{\rho_0c_0}\,L}\frac{dt}{d\alpha}+2\int_0^t\frac{(n_kU_k+c_0)\,dt}{\sqrt{(U_k+c_0n_k)^2}\sqrt{\rho_0c_0}\,L}+\frac{4}{\gamma+1}f'(\alpha)=0$$

From this we deduce that the law of variation of α at the head of the shock wave, along the ray, is

$$\alpha^2 \int_{l_0}^{l} \frac{(n_k U_k + c_0)\, dl}{(U_k + c_0 n_k)^2 \sqrt{\rho_0 c_0 L}} = \frac{4}{\gamma + 1} \int_{\alpha}^{\alpha_0} \alpha f'(\alpha)\, d\alpha \tag{2.7}$$

If at the initial instant the pressure across the wave is continuous, the moment when the discontinuity appears is determined from equation (2.5) in the same way as in the case of the one-dimensional flow of gas (6). For $l >> l_0$ the disturbed region, from the condition of conservation of energy, must contain both a region with $\Delta > 0$, and a region with $\Delta < 0$ (this condition does not apply to a plane wave). From equations (2.5) and (2.7) it follows that for $l >> l_0$, the pressure profile behing the wave front is approximately linear, and independent of the profile of the wave at the initial instant. The amplitude of the wave with linear pressure profile behind the front changes, from equation (2.7), according to the law

$$\Delta = \frac{\alpha_0 \sqrt{\rho_0 c_0}}{L} \left(1 + \frac{\gamma + 1}{2} \frac{\alpha_0}{\lambda_0} \int_{l_0}^{l} \frac{(n_k U_k + c_0)\, dl}{(U_k + c_0 n_k)^2 \sqrt{\rho_0 c_0 L}}\right)^{-1/2}$$

where λ_0 is the breadth of the region $\Delta > 0$ at $t = 0$.

In a uniform medium at rest the surface of the wave front tends to become spherical as $R \to \infty$, and the amplitude of the wave falls off according to the law $\Delta = B/R \sqrt{\ln R}$ where $B =$ constant for the given ray. The quantity B in this relation may be different in different directions, if the initial conditions do not have spherical symmetry. This solution for the uniform stationary medium is the asymptotic solution of equation (1.1) as $R \to \infty$, for an arbitrary disturbance of finite energy, although in this case the breadth of the disturbed region is compared only with the radius of the curved wave front and $\lambda/R \to 0$ as $R \to \infty$.

For a nonuniform moving medium the condition

$$H \gg \lambda = \lambda_0 + \frac{\gamma + 1}{4} \int_0^t \frac{\Delta}{\rho_0 c_0} \frac{n_k U_k + c_0}{\sqrt{(U_k + c_0 n_k)^2}}\, dt$$

must also be used, where the integral is evaluated along the path of the front. This fact limits the application of the resulting solution when $l \to \infty$.

BIBLIOGRAPHY

1. Keller, J.B., Geometrical acoustics. I. The theory of weak shock waves. *J. Appl. Phys.* Vol. 25, No. 8, 1954.

2. Landau, L.D. and Lifshitz, E.M., *Mekhanika sploshnikh sred (Mechanics of Continuous Media)*. GITTL, 1953.

3. Landau, L.D., Ob udarnykh volnakh na dalekikh rastoianiiakh ot mesta ikh vozniknoveniia (On shock waves at great distances from their source). *PMM* Vol. 9, No. 4, 1945.

4. Khristianovich, S.A., Udarnaia volna no znachitel'nom rasstoianii ot mesta vzryva (Shock waves at great distances from the origin of an explosion). *PMM* Vol. 20, No. 5, 1956.

5. Whitham, G.B., The propagation of spherical blast. *Proc. Roy. Soc.* A Vol. 203, No. 1075, 1950.

6. Courant, R. and Friedrichs, K., *Supersonic Flow and Shock Waves.* Interscience, New York, 1948.

Translated by D.T.W.